J. DIFFERENTIAL GEOMETRY
52 (1999) 145-171

LAGRANGIAN TWO-SPHERES CAN BE SYMPLECTICALLY KNOTTED

PAUL SEIDEL

1. Introduction

In the past few years there have been several striking results about the topology of Lagrangian surfaces in symplectic four-manifolds. The general tendency of these results is that many isotopy classes of embedded surfaces do not contain Lagrangian representatives. This is called the *topological unknottedness* of Lagrangian surfaces; see [4] for a survey. The aim of this paper is to complement this picture by showing that Lagrangian surfaces can be *symplectically knotted* in infinitely many inequivalent ways. That is to say, a single isotopy class of embedded surfaces can contain infinitely many Lagrangian representatives which are non-isotopic in the Lagrangian sense. The symplectic four-manifolds for which we prove this are non-compact in a mild sense; they are interiors of compact symplectic manifolds with contact type boundary. For instance, one can take

$$M = \{z \in \mathbb{C}^3 \mid |z| < R, \quad z_1^2 + z_2^2 = z_3^{m+1} + \tfrac{1}{2}\} \tag{1.1}$$

for $m \geq 3$ and large R, with the standard symplectic form. It seems likely that the same phenomenon occurs for a large class of closed symplectic four-manifolds. At present, the best result in this direction is that for any N, there is a closed four-manifold containing N Lagrangian two-spheres which are all isotopic as smooth submanifolds but pairwise

Received March 30, 1998, and, in revised form, May 28, 1999. The author was supported by the European Community and by NSF grant DMS 9304580.

non-isotopic as Lagrangian submanifolds. For example, the smooth hypersurface in $\mathbb{CP}^3$ of degree $N+4$ has this property. In this paper we will not prove the statement about closed manifolds.

As a by-product of our construction it turns out that the four-manifolds which we consider have a symplectic automorphism ϕ with the following property: ϕ is isotopic to the identity as a diffeomorphism, but not symplectically so. Moreover, none of the iterates ϕ^r are symplectically isotopic to the identity. To be precise, ϕ is the identity outside a compact subset, and it can be deformed to the identity map inside the group of diffeomorphisms with this property. In contrast, it cannot be deformed to the identity through symplectomorphisms even if we allow these to behave arbitrarily at infinity; and the same holds for ϕ^r. Just as in the case of Lagrangian two-spheres, one can obtain a weaker form of this statement for closed symplectic four-manifolds; see [19]. Kronheimer has obtained results of a related kind using a parametrized version of Seiberg-Witten theory [9].

We recall some basic definitions. Let M be a differentiable manifold. A differentiable isotopy between two compact submanifolds $L_0, L_1 \subset M$ is a compact submanifold $\widetilde{L} \subset M \times [0;1]$ with $\widetilde{L} \cap (M \times \{t\}) = L_t \times \{t\}$ for $t = 0, 1$ and such that the intersection $\widetilde{L} \cap (M \times \{t\})$ is transverse for all $t \in [0;1]$. If (M, ω) is a symplectic manifold, a Lagrangian isotopy between two compact Lagrangian submanifolds $L_0, L_1 \subset M$ is an $\widetilde{L}$ as above, with the additional property that $\widetilde{L} \cap (M \times \{t\}) \subset M \times \{t\}$ is Lagrangian for all t. This means that the pullback of ω to $\widetilde{L}$ via the projection $\widetilde{L} \subset M \times [0;1] \longrightarrow M$ is of the form $\theta \wedge dt$ for some $\theta \in \Omega^1(\widetilde{L})$. The Lagrangian isotopy $\widetilde{L}$ is called exact if one can write $\theta \wedge dt = d(H\,dt)$ for some $H \in C^\infty(\widetilde{L}, \mathbb{R})$. Any Lagrangian isotopy between Lagrangian submanifolds with vanishing first Betti number is exact.

Two diffeomorphisms $\phi_0, \phi_1 : M \longrightarrow M$ are differentiably isotopic if they can be joined by a path $(\phi_t)_{0 \leq t \leq 1}$ in $\mathrm{Diff}(M)$ which is smooth (in the sense that $F(x,t) = \phi_t(x)$ is a smooth map from $M \times [0;1]$ to M). Two compact submanifolds L_0, L_1 which are differentiably isotopic are also ambient isotopic, that is, there is a path (ϕ_t) with $\phi_0 = \mathrm{id}$ and $\phi_1(L_0) = L_1$. Similarly, one can define the notion of symplectic isotopy between two symplectic automorphisms, and two compact Lagrangian submanifolds which are exact Lagrangian isotopic are also ambient isotopic in the symplectic sense.

We now explain how our examples of symplectically knotted Lagrangian two-spheres are constructed. Given a Lagrangian two-sphere L in a symplectic four-manifold (M, ω), one can define a symplectic automorphism τ_L of M called the *generalized Dehn twist along* L. This automorphism is trivial outside a tubular neighbourhood of L, and its restriction to L is the antipodal involution of S^2. The definition of τ_L involves additional choices, but the outcome is independent of these choices up to symplectic isotopy. The topological analogues of generalized Dehn twists are certain diffeomorphisms associated to embedded two-spheres of self-intersection -2 in smooth four-manifolds. These maps are familiar to topologists. The symplectic viewpoint appears for the first time in Arnol'd's paper [1].

We will use these generalized Dehn twists in the following way: assume that M contains two Lagrangian two-spheres L_1, L_2. Then one can construct an infinite family of such two-spheres

$$L_1^{(r)} = \tau_{L_2}^{2r}(L_1) \quad (r \in \mathbb{Z}) \tag{1.2}$$

by twisting L_1 around L_2. We have used only even iterates of τ_{L_2} because the square of any generalized Dehn twist is differentiably isotopic to the identity (this was explained to the author by Peter Kronheimer). As a consequence all the $L_1^{(r)}$ are isotopic as differentiable submanifolds. Our main result is that they are not always Lagrangian isotopic.

Theorem 1.1. *Let (M, ω) be the interior of a compact symplectic manifold with contact type boundary. We assume that (M, ω) contains three Lagrangian two-spheres L_1, L_2, L_3 such that $L_1 \cap L_3 = \emptyset$, and such that both $L_1 \cap L_2$ and $L_2 \cap L_3$ are transverse and consist of a single point (this is called an (A_3)-configuration of Lagrangian two-spheres). Moreover, we require that both $[\omega] \in H^2(M; \mathbb{R})$ and $c_1(M, \omega) \in H^2(M; \mathbb{Z})$ vanish. Let $L_1^{(r)}$ be as in (1.2). Then $L_1^{(s)}$ and $L_1^{(t)}$ are not Lagrangian isotopic unless $s = t$.*

The assumptions are very restrictive. To provide some concrete examples, we will prove that the affine hypersurfaces (1.1) satisfy them. As an immediate consequence, one obtains the following result which establishes our second claim:

Corollary 1.2. *In the situation of Theorem 1.1 no iterate of $\tau_{L_2}^2$ is symplectically isotopic to the identity.*

Acknowledgements. I am indebted to Peter Kronheimer for several helpful conversations, and to Leonid Polterovich for introducing me to Lagrangian surgery. Parts of this paper were written during visits to Tel Aviv University and Stanford University.

2. An outline of the proof

Let (M, ω) be the interior of a compact symplectic four-manifold with contact type boundary, and assume that $[\omega] \in H^2(M; \mathbb{R})$ is zero. The Floer homology $HF(L, L')$ is a finite-dimensional $\mathbb{Z}/2$-vector space associated to any pair (L, L') of Lagrangian two-spheres in M. It is invariant under Lagrangian isotopy in the following sense: if L'' is Lagrangian isotopic to L, then $HF(L, L') \cong HF(L'', L')$ for all L'. Conversely, to prove that two Lagrangian two-spheres L and L'' are not Lagrangian isotopic, it is sufficient to find a third two-sphere L' such that $HF(L, L') \not\cong HF(L'', L')$. The main difficulty is that the Floer homology groups are difficult to compute. If L and L' are disjoint, $HF(L, L') = 0$. Floer [7] proved that if L and L' are Lagrangian isotopic, then $HF(L, L') \cong H_*(L; \mathbb{Z}/2)$. We will use a generalization of Floer's result due to Pozniak [14].

Definition 2.1. $L, L' \subset M$ have *clean intersection* if $N = L \cap L'$ is a smooth submanifold of M and satisfies $TN = (TL|N) \cap (TL'|N)$.

To simplify the statement, we consider only the case where (in addition to the conditions imposed above) the first Chern class of M vanishes. Then the Floer homology of (L, L') is a graded group. The grading is not quite unique. However, this is irrelevant for our purpose since we use it only as a computational device: ultimately, only the ungraded Floer homology group will serve as an invariant. Pozniak's result can be formulated as follows: if L and L' have clean intersection, there is a spectral sequence which converges to $HF_*(L, L')$ and whose E^1-term is

$$E^1_{pq} = \begin{cases} H_{p+q-i'(C_p)}(C_p; \mathbb{Z}/2), & 1 \leq p \leq r, \\ 0, & \text{otherwise.} \end{cases}$$

Here $C_1, \dots, C_r$ are the connected components of $L \cap L'$ ordered in a way determined by the action functional. $i'(C_p) \in \mathbb{Z}$ is a kind of Maslov index. This is a homology spectral sequence, that is, the d-th differential $(d \geq 1)$ has degree $(-d, d-1)$.

In the situation of Theorem 1.1 one can arrange that for a given $r > 0$, $L_1^{(r)}$ intersects L_3 cleanly in r circles $C_1, \dots, C_r$. With the right ordering one finds that $i'(C_p) = 2p$. Hence the E^1 term of the Pozniak spectral sequence for $(L_1^{(r)}, L_3)$ is

$$E^1_{pq} = \begin{cases} \mathbb{Z}/2, & 1 \leq p \leq r \text{ and } q = p \text{ or } p+1, \\ 0, & \text{otherwise.} \end{cases}$$

Since the entry $E^1_{11} = \mathbb{Z}/2$ survives to E^∞, we have $HF(L_1^{(r)}, L_3) \neq 0$. On the other hand, $HF(L_1, L_3) = 0$ because L_1 and L_3 are assumed to be disjoint. Hence L_1 is not Lagrangian isotopic to $L_1^{(r)}$, for any $r > 0$. It follows that $L_1^{(s)}$ is not Lagrangian isotopic to $L_1^{(t)}$ whenever $s < t$, because otherwise $L_1^{(t-s)}$ would be Lagrangian isotopic to L_1.

3. Floer homology and clean intersection

This section and the next two contain a more detailed account of Floer homology and of Pozniak's results. Our definition of Floer homology is essentially Floer's original one [5]. The construction has been generalized by Oh [11] but this generalization is unnecessary for our purpose.

Let $(\overline{M}, \bar{\omega})$ be a compact symplectic manifold with contact type boundary, and (M, ω) its interior. Fix an $\bar{\omega}$-compatible almost complex structure $\overline{J}$ which makes the boundary $\overline{J}$-convex.

Let $L, L' \subset M$ be a pair of compact Lagrangian submanifolds, and $\mathcal{P}(L, L')$ the space of smooth paths $\gamma : I = [0;1] \longrightarrow M$ such that $\gamma(0) \in L$ and $\gamma(1) \in L'$. Let

$$\alpha(\gamma)\xi = \int_I \omega(\frac{d\gamma}{dt}, \xi(t))\, dt$$

be the action one-form on $\mathcal{P}(L, L')$. α is always closed. The Floer homology group $HF(L, L')$ is defined whenever $[\alpha] \in H^1(\mathcal{P}(L, L'); \mathbb{R})$ is zero. If this holds for (L, L'), it also holds for (L'', L') whenever L'' is exact Lagrangian isotopic to L, and then $HF(L'', L') \cong HF(L, L')$. More precisely, an exact Lagrangian isotopy from L to L'' determines an isomorphism between the two Floer homologies. If we assume that $[\omega] = 0$ and $H^1(L; \mathbb{R}) = H^1(L'; \mathbb{R}) = 0$, then $[\alpha] = 0$ and all Lagrangian isotopies are exact. In this way one recovers the description of Floer homology given in the previous section.

We will now review briefly the definition of $HF(L, L')$. By assumption, one can choose a function $a : \mathcal{P}(L, L') \longrightarrow \mathbb{R}$ with $da = \alpha$. For any $H \in \mathcal{H} = C_c^\infty(I \times M, \mathbb{R})$, let

$$a_H(\gamma) = a(\gamma) + \int_I H_t(\gamma(t))\, dt.$$

Let $Z(H) \subset \mathcal{P}(L, L')$ be the set of critical points of a_H. For $H = 0$ the critical points are the constant paths γ_x at $x \in L \cap L'$. In general, γ is a critical point iff it is an orbit of the flow (ϕ_t^H) induced by H. Hence $Z(H)$ can be identified naturally with $\phi_1^H(L) \cap L'$. Let $\mathcal{H}^{\mathrm{reg}} \subset \mathcal{H}$ be the dense subset of those H for which $\phi_1^H(L)$ and L' intersect transversely. For $H \in \mathcal{H}^{\mathrm{reg}}$ one defines $CF(H)$ to be the $\mathbb{Z}/2$-vector space generated by the finite set $Z(H)$. Let $\mathcal{J}$ be the space of one-parameter families $\mathbf{J} = (J_t)_{t \in I}$ of almost complex structures on M which have the following properties:

(1) Every J_t is ω-compatible.

(2) There is a neighbourhood $U \subset \overline{M}$ of $\partial \overline{M}$ such that $J_t | U \cap M = \overline{J} | U \cap M$ for all t.

For $H \in \mathcal{H}$, $\mathbf{J} \in \mathcal{J}$, and $\gamma_-, \gamma_+ \in Z(H)$, we denote by $\mathcal{M}_{H,\mathbf{J}}(\gamma_-, \gamma_+)$ the set of smooth maps $u : \mathbb{R} \times I \longrightarrow M$ such that

$$\begin{cases} u(s,0) \in L, \quad u(s,1) \in L', \quad \lim_{s \to \pm\infty} u(s, \cdot) = \gamma_\pm, \\ \frac{\partial u}{\partial s} + J_t(u) \left(\frac{\partial u}{\partial t} - X_t^H(u) \right) = 0. \end{cases} \tag{3.1}$$

Here X^H is the (time-dependent) Hamiltonian vector field of H. If one thinks of u as a map $\mathbb{R} \longrightarrow \mathcal{P}(L, L')$, the solutions of (3.1) are the bounded negative gradient flow lines of a_H with respect to an L^2-metric defined by $\mathbf{J}$. Hence $\mathcal{M}_{H,\mathbf{J}}(\gamma_-, \gamma_+)$ can be nonempty only if $a_H(\gamma_-) > a_H(\gamma_+)$ or $\gamma_- = \gamma_+$. The assumption on the behaviour of $\mathbf{J}$ at infinity implies that the union of the images of all solutions of (3.1) lies inside a compact subset $K \subset M$. This removes any possible problems arising from the non-compactness of M.

Assume that $H \in \mathcal{H}^{\mathrm{reg}}$, and let $\mathcal{J}^{\mathrm{reg}}(H) \subset \mathcal{J}$ be the subspace of those $\mathbf{J}$ for which all solutions of (3.1) are reguar. Regularity is defined as the surjectivity of the linearization of (3.1) in suitable Sobolev spaces. $\mathcal{J}^{\mathrm{reg}}(H)$ is a dense subset; the necessary transversality arguments were carried out in [8] and [10]. If $\mathbf{J}$ is in $\mathcal{J}^{\mathrm{reg}}(H)$, the spaces $\mathcal{M}_{H,\mathbf{J}}(\gamma_-, \gamma_+)$ have a natural structure of finite-dimensional smooth manifolds. Moreover, any one of them has only finitely many one-dimensional connected components. Let $n_{H,\mathbf{J}}(\gamma_-,\gamma_+) \in \mathbb{Z}/2$ be the number mod 2 of these components. Floer proved that the homomorphism

$$\begin{aligned}&\partial(H,\mathbf{J}) : CF(H) \longrightarrow CF(H),\\ &\partial(H,\mathbf{J})\langle\gamma_-\rangle = \sum_{\gamma_+\in Z(H)} n_{H,\mathbf{J}}(\gamma_-,\gamma_+)\langle\gamma_+\rangle\end{aligned}$$

satisfies $\partial(H,\mathbf{J})\circ\partial(H,\mathbf{J}) = 0$. Floer homology is defined by

$$HF(L,L',H,\mathbf{J}) = \ker\partial(H,\mathbf{J})/\mathrm{im}\;\partial(H,\mathbf{J}).$$

A continuation argument proves that this is independent of the choice of $\mathbf{J}$ and H up to canonical isomorphisms. Therefore one can omit $(H,\mathbf{J})$ from the notation. By a simple change of variables, the independence of H implies the invariance under exact Lagrangian isotopy. A detailed exposition of the continuation argument (in a slightly different context) can be found in [18].

Floer [5] introduced versions of Floer homology which are local with respect to certain parts of $N = L\cap L'$. More precisely, let $C\subset N$ be a path component which is both open and closed in N. Choose a $\mathbf{J}_0\in\mathcal{J}$ and an open neighbourhood $U\subset M$ of C such that $N\cap U = C$. There is a contractible neighbourhood $\mathcal{V}\subset\mathcal{H}\times\mathcal{J}$ of $(0,\mathbf{J}_0)$ such that any $(H,\mathbf{J})\in\mathcal{V}$ has the following properties:

(1) any $\gamma\in Z(H)$ such that $\mathrm{im}(\gamma)\subset\overline{U}$ satisfies $\mathrm{im}(\gamma)\subset U$;

(2) if $\gamma_-,\gamma_+\in Z(H)$ satisfy $\mathrm{im}(\gamma_\pm)\subset U$, then any $u\in\mathcal{M}_{H,\mathbf{J}}(\gamma_-,\gamma_+)$ has $\mathrm{im}(u)\subset U$.

This can be proved by a simple limit argument. Now choose $(H,\mathbf{J})\in\mathcal{V}$ such that $H\in\mathcal{H}^{\mathrm{reg}}$ and $\mathbf{J}\in\mathcal{J}^{\mathrm{reg}}(H)$. Let $CF^{\mathrm{loc}}(H;C)$ be the $\mathbb{Z}/2$-vector space generated by those $\gamma\in Z(H)$ with $\mathrm{im}(\gamma)\subset U$, and

$$\partial^{\mathrm{loc}}(H,\mathbf{J};C) : CF^{\mathrm{loc}}(H;C) \longrightarrow CF^{\mathrm{loc}}(H;C)$$

the homomorphism obtained by considering only those $u\in\mathcal{M}_{H,\mathbf{J}}(\gamma_-,\gamma_+)$ such that $\mathrm{im}(u)\subset U$. $\partial^{\mathrm{loc}}(H,\mathbf{J};C)^2 = 0$, and one defines

$$HF^{\mathrm{loc}}(L,L',H,\mathbf{J};C) = \ker\partial^{\mathrm{loc}}(H,\mathbf{J};C)/\mathrm{im}\;\partial^{\mathrm{loc}}(H,\mathbf{J};C).$$

One can prove that the local Floer homology is independent of the choice of $(H,\mathbf{J})\in\mathcal{V}$. The proof is again by a continuation method.

Recall that in arguments of this kind one studies maps $u : \mathbb{R} \times I \longrightarrow M$ which satisfy an equation

$$\begin{cases} u(s,0) \in L, \quad u(s,1) \in L', \\ \frac{\partial u}{\partial s} + J_{s,t}(u)\left(\frac{\partial u}{\partial t} - X_{s,t}(u)\right) = 0, \end{cases} \tag{3.2}$$

with suitable asymptotic behaviour. Here $J_{s,t}$ is a two-parameter family of almost complex structures, and $X_{s,t}$ is the family of Hamiltonian vector fields on M induced by a function $K \in C^\infty(\mathbb{R} \times I \times M, \mathbb{R})$. In the case of local Floer homology, one considers such families with the property that $H_s = K_{s,\cdot} \in \mathcal{H}$ and $\mathbf{J}_s = J_{s,\cdot}$ satisfy $(H_s, \mathbf{J}_s) \in \mathcal{V}$ for all s. In itself this does not imply that any solution of (3.2) with limits in U satisfies $\mathrm{im}(u) \subset U$. However, a limit argument shows that this holds if the path $s \longmapsto (H_s, \mathbf{J}_s) \in \mathcal{V}$ is close to a constant path in a suitable sense. This is sufficient to prove that the local Floer homology is independent of H and $\mathbf{J}$. The same argument also proves that it remains unchanged under small variations of $\mathbf{J}_0$, and hence is independent of $\mathbf{J}_0$. Therefore one obtains a well-defined group $HF^{\mathrm{loc}}(L, L'; C)$. This group is called local because it depends only on the behaviour of L and L' in an arbitrarily small neighbourhood of C. For instance, if $C = \{x\}$ is a transverse intersection point of L and L', then $HF^{\mathrm{loc}}(L, L'; C) \cong \mathbb{Z}/2$.

Theorem 3.1 (Pozniak [14, Theorem 3.4.11]). *Assume that L and L' intersect cleanly along C. Then $HF^{\mathrm{loc}}(L, L'; C)$ is isomorphic to the total homology $H_*(C; \mathbb{Z}/2)$.*

This is the main result of [14]. We will not reproduce Pozniak's proof here. Instead, we will give (at the end of this section) an indirect proof of a special case of Theorem 3.1:

Proposition 3.2. *Assume that (M, ω) is four-dimensional, that L and L' are orientable, and that C is a circle along which L and L' intersect cleanly. Then $HF^{\mathrm{loc}}(L, L'; C) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$.*

The ordinary Floer homology and its local version are related in the following way: let $L, L' \subset M$ be two compact Lagrangian submanifolds such that $N = L \cap L'$ can be decomposed into finitely many path components $C_1, \dots, C_r$, each of which is open (and closed) in N. As always, we assume that $[\alpha] \in H^1(\mathcal{P}(L, L'); \mathbb{R})$ vanishes. Let $a_j = a(\gamma_{x_j})$ where γ_{x_j} is the constant path at a point $x_j \in C_j$. We assume that the C_j have been ordered in such a way that $a_1 \leq a_2 \leq \cdots \leq a_r$. Choose a $\mathbf{J}_0 \in \mathcal{J}$ and open neighbourhoods $U_j \subset M$ of C_j whose closures are pairwise disjoint. There is a contractible neighbourhood $\mathcal{V} \subset \mathcal{H} \times \mathcal{J}$ of $(0, \mathbf{J}_0)$ such that any $(H, \mathbf{J}) \in \mathcal{V}$ has the following properties:

(1) any $\gamma \in Z(H)$ satisfies $\mathrm{im}(\gamma) \subset U_j$ for some j;

(2) if $\gamma_-, \gamma_+ \in Z(H)$ satisfy $\mathrm{im}(\gamma_-) \subset U_j$ and $\mathrm{im}(\gamma_+) \subset U_k$ with $j < k$, then $\mathcal{M}_{H,\mathbf{J}}(\gamma_-, \gamma_+) = \emptyset$;

(3) for $\gamma_-, \gamma_+ \in Z(H)$ such that $\mathrm{im}(\gamma_\pm) \subset U_j$ every $u \in \mathcal{M}_{H,\mathbf{J}}(\gamma_-, \gamma_+)$ satisfies $\mathrm{im}(u) \subset U_j$.

Now take $(H, \mathbf{J}) \in \mathcal{V}$ such that $H \in \mathcal{H}^{\mathrm{reg}}$ and $\mathbf{J} \in \mathcal{J}^{\mathrm{reg}}(H)$. Consider the filtration of $CF(H)$ by the subspaces $CF(H)^{[j]}$ generated by those $\gamma \in Z(H)$ such that $\mathrm{im}(\gamma) \subset U_1 \cup \cdots \cup U_j$. It is a consequence of the properties which we have just stated that $\partial(H, \mathbf{J})$ preserves this filtration, and that the homology of the induced boundary operator on $CF(H)^{[j]}/CF(H)^{[j-1]}$ is isomorphic to the local Floer homology $HF^{\mathrm{loc}}(L, L'; C_j)$.

Proof of Proposition 3.2. The first step in the proof is a local normal form theorem [14, Proposition 3.4.1] for cleanly intersecting Lagrangian submanifolds. In our case this says that one can identify a neighbourhood of C in M symplectically with a neighbourhood of $S^1 \times 0$ in $(S^1 \times \mathbb{R}^3, ds \wedge dx_1 + dx_2 \wedge dx_3)$ in such a way that L is mapped to $S^1 \times 0 \times \mathbb{R} \times 0$ and L' is mapped to $S^1 \times 0 \times 0 \times \mathbb{R}$. In particular, the local behaviour of L and L' near C is the same in all cases covered by Proposition 3.2. Hence the local Floer homology group is also the same in all cases. We denote this group, which we want to compute, by G.

The second step is to show that G is either 0 or $\mathbb{Z}/2 \oplus \mathbb{Z}/2$. To do this, one observes that, given a neighbourhood $U \subset M$ of C, there are arbitrarily small $H \in \mathcal{H}$ such that there are precisely two $\gamma \in Z(H)$ with $\mathrm{im}(\gamma) \subset U$. In the local model $S^1 \times \mathbb{R}^3$, one can take

$$H_t(z, x_1, x_2, x_3) = h(z)\psi(|x_1|^2 + |x_2|^2 + |x_3|^2), \tag{3.3}$$

where h is a Morse function on S^1 with two critical points, and ψ is a cutoff function ($\psi(r) = 0$ for large r and $= 1$ for small r).

The final step is to exclude the possibility that $G = 0$. Let $M = T^*T^2$ with the standard symplectic structure. Take a Morse-Bott function $k \in C^\infty(T^2, \mathbb{R})$ whose critical set consists of two circles. Let $L \subset M$ be the zero-section and $L' \subset M$ the graph of dk. Then L' and L intersect cleanly in two circles. The considerations above show that for suitable H and $\mathbf{J}$, the chain group $CF(H)$ and the boundary operator $\partial(H,\mathbf{J})$ have the following form: there is a subgroup $CF(H)^{[1]}$ which is preserved by $\partial(H,\mathbf{J})$. The homology of this subgroup is equal to G, and the homology of the quotient is also equal to G. If we assume that G is zero, the long exact sequence would imply that $HF(L,L')=0$. However, since L' is exact Lagrangian isotopic to L, Floer's theorem says that $HF(L,L')\cong H_*(T^2;\mathbb{Z}_2)$. q.e.d.

The proof of Proposition 3.2 was based on the relationship between clean intersection in cotangent bundles and Morse-Bott functions. In fact, one can see Pozniak's approach as an analogue of the Morse-Bott spectral sequence in ordinary homology (see [2] for an exposition). Other arguments in Floer homology based on the same principles can be found in [17] and [12].

4. The grading on Floer homology

The material collected in this section is due to Viterbo [21], Floer [6] and Robbin-Salamon [15] [16].

Let $\mathcal{L}(n)$ be the Lagrangian Grassmannian, which parametrizes linear Lagrangian subspaces in $\mathbb{R}^{2n}$. The Maslov index associates an integer $\mu(\lambda,\lambda')$ to a pair of loops $\lambda,\lambda':S^1\longrightarrow\mathcal{L}(n)$. This index is invariant under homotopy and under conjugation of both λ and λ' by a loop in $\mathrm{Sp}(2n;\mathbb{R})$. Usually, one considers the Maslov index as an invariant of a single loop in $\mathcal{L}(n)$; this corresponds to taking λ' to be a constant loop. Let $L,L'\subset(M^{2n},\omega)$ be two Lagrangian submanifolds. A loop in $\mathcal{P}(L,L')$ is a map $u:S^1\times I\longrightarrow M$ with boundary values in L resp. L'. After choosing a symplectic trivialization of u^*TM one obtains two loops $\lambda(s)=TL_{u(s,0)}$, $\lambda'(s)=TL'_{u(s,1)}$ in $\mathcal{L}(n)$. The Maslov indices of such loops determine a class $\chi\in H^1(\mathcal{P}(L,L');\mathbb{Z})$.

The *Maslov index for paths* assigns a half-integer $\mu(\lambda,\lambda')\in\frac{1}{2}\mathbb{Z}$ to any pair of paths $\lambda,\lambda':[a;b]\longrightarrow\mathcal{L}(n)$. It is a generalization of the ordinary Maslov index, to which it reduces if both paths are closed, and has the following basic properties:

- (i) $\mu(\lambda,\lambda')$ depends on λ,λ' only up to homotopy with fixed endpoints.
- (ii) The Maslov index remains the same if one conjugates both λ and λ' by a path $\Psi:[a;b]\longrightarrow\mathrm{Sp}(2n,\mathbb{R})$.
- (iii) μ is additive under concatenation (of pairs of paths).

(iv) $\mu(\lambda, \lambda') = -\mu(\lambda', \lambda)$.
(v) $\mu(\lambda, \lambda')$ vanishes if the dimension of $\lambda(s) \cap \lambda'(s)$ is constant.
(vi) $\mu(\lambda, \lambda') \equiv \frac{1}{2}\dim(\lambda(a) \cap \lambda'(a)) - \frac{1}{2}\dim(\lambda(b) \cap \lambda'(b)) \mod 1$.

Take a path

$$[a;b] \longrightarrow \mathcal{P}(L, L')$$

from γ_{x_-} to γ_{x_+}, where

$$x_-, x_+ \in L \cap L'.$$

Such a path is given by a map $u : [a;b] \times I \longrightarrow M$ with suitable boundary conditions. Let $E = u^*(TM, \omega)$. Choose a Lagrangian subbundle $F \subset E$ such that $F_{(a,t)} = TL_{x_-}$ and $F_{(b,t)} = TL_{x_+}$ for all t, and $F_{(s,0)} = TL_{u(s,0)}$ for all s. After choosing a trivialization of E, one obtains two paths $\lambda, \lambda' : [a;b] \longrightarrow \mathcal{L}(n)$, namely $\lambda(s) = F_{(s,1)}$ and $\lambda'(s) = TL'_{u(s,1)}$. Properties (i) and (ii) ensure that

$$I(u) \stackrel{\text{def}}{=} \mu(\lambda, \lambda')$$

is independent of the trivialization and of the choice of F. $I(u)$ depends on u only up to homotopies which keep the endpoints $\gamma_{x_\pm}$ fixed. It is also additive under concatenation. Moreover, if u and u' are two paths with the same endpoints, one has $I(u) - I(u') = \chi(v)$, where $\chi \in H^1(\mathcal{P}(L, L'); \mathbb{Z})$ is the class defined above, and v is the loop in $\mathcal{P}(L, L')$ obtained by gluing u and u' at both endpoints. It follows that if $\chi = 0$ one can find numbers $i(\gamma_x) \in \frac{1}{2}\mathbb{Z}$ for every $x \in L \cap L'$, such that

$$I(u) = i(\gamma_{x_-}) - i(\gamma_{x_+})$$

for every path u from γ_{x_-} to γ_{x_+}. Because of property (vi) one can also arrange that

$$i(\gamma_x) \equiv \tfrac{1}{2}\dim(TL_x \cap TL'_x) \mod 1. \tag{4.1}$$

Numbers $i(\gamma_x)$ with these two properties are called a *coherent choice of indices* for (L, L').

Take $H_-, H_+ \in \mathcal{H}$, and let $v : [a;b] \longrightarrow \mathcal{P}(L, L')$ be a path from a point $\gamma_- \in Z(H_-)$ to a point $\gamma_+ \in Z(H_+)$. A slight extension of the construction above associates to such a path a number $I_{H_-,H_+}(v) \in \frac{1}{2}\mathbb{Z}$.

The details are as follows: choose a trivialization of $E = v^*TM$ and a Lagrangian subbundle $F \subset E$ such that

$$F_{(a,t)} = \phi_t^{H_-}(TL_{\gamma_-(0)}), \quad F_{(b,t)} = \phi_t^{H_+}(TL_{\gamma_+(0)}), \quad F_{(s,0)} = TL_{v(s,0)}.$$

Again, one obtains two paths in $\mathcal{L}(n)$: $\lambda(s) = F_{(s,1)}$ and $\lambda'(s) = TL'_{v(s,1)}$. $I_{H_-,H_+}(v)$ is defined as the Maslov index of these paths. If $\chi = 0$, one can find numbers $i_H(\gamma) \in \frac{1}{2}\mathbb{Z}$ for any $H \in \mathcal{H}$ and $\gamma \in Z(H)$, such that

$$I_{H_-,H_+}(v) = i_{H_-}(\gamma_-) - i_{H_+}(\gamma_+)$$

for all v, H_-, H_+, and

$$i_H(\gamma) \equiv \tfrac{1}{2}\dim(D\phi_1^H(TL_{\gamma(0)}) \cap TL'_{\gamma(1)}) \mod 1. \tag{4.2}$$

Moreover, given a coherent choice of indices $i(\gamma_x)$, one can choose the $i_H(\gamma)$ in such a way that $i_H(\gamma_x) = i(\gamma_x)$ for $H = 0$.

Now let (M, ω) be the interior of a compact symplectic manifold with contact type boundary. We assume that L, L' are compact, and that $[\alpha] \in H^1(\mathcal{P}(L, L'); \mathbb{R})$ and $\chi \in H^1(\mathcal{P}(L, L'); \mathbb{Z})$ vanish. Fix a coherent choice of indices for (L, L'), and extend that choice to more general numbers $i_H(\gamma)$ as above. For $H \in \mathcal{H}^{\mathrm{reg}}$ and $k \in \mathbb{Z}$, let $CF_k(H) \subset CF(H)$ be the subgroup generated by those $\gamma \in Z(H)$ such that $i_H(\gamma) = k$; it follows from (4.2) that $i_H(\gamma)$ is always integral if $H \in \mathcal{H}^{\mathrm{reg}}$. Choose a $\mathbf{J} \in \mathcal{J}^{\mathrm{reg}}(H)$. An index theorem due to Floer [6] shows that $\partial(H, \mathbf{J})$ has degree -1 with respect to the grading of $CF(H)$ which we have introduced. Hence one obtains a grading of $HF(L, L', H, \mathbf{J})$. This grading is compatible with the canonical isomorphisms between these groups for different $(H, \mathbf{J})$.

One case when χ vanishes is when the first Chern class of (M, ω) is zero and $H^1(L) = H^1(L') = 0$. This shows that the grading of Floer homology exists in the situation described in section 2.

Clearly, a choice of grading for $HF(L, L')$ also induces a grading of all local Floer homology groups. As in the previous section, assume that $N = L \cap L'$ has finitely many path components $C_1, \dots, C_r$ which are open in N. Then one obtains a filtration of the chain complex $(CF_*(H), \partial(H, \mathbf{J}))$, for suitable $(H, \mathbf{J})$, and the homology of successive quotients is the local Floer homology $HF_*^{\mathrm{loc}}(L, L'; C_j)$. Therefore there is a spectral sequence which converges to $HF_*(L, L')$, with

$$E^1_{pq} = HF^{\mathrm{loc}}_{p+q}(L, L'; C_p). \tag{4.3}$$

Now assume that L and L' have clean intersection. It follows from property (v) that for any coherent choice of indices the function $x \longrightarrow i(\gamma_x)$ is locally constant on $L \cap L'$. Let $i(C_j)$ be the value of this function on C_j, and $i'(C_j) = i(C_j) - \frac{1}{2} \dim C_j$ (equation (4.1) implies that the $i'(C_j)$ are integral). Theorem 3.1 has the following graded version:

Theorem 4.1. $HF_*^{\mathrm{loc}}(L, L'; C) \cong H_{*-i'(C)}(C; \mathbb{Z}/2)$.

Given this, one obtains the spectral sequence used in section 2 as a special case of (4.3). We will not prove Theorem 4.1 but only the case corresponding to Proposition 3.2. To do this, introduce local coordinates around C as in the proof of that Proposition, and take H as in (3.3). If h is sufficiently small, the subset of $Z(H)$ which consists of paths near C contains only the constant paths $\gamma_{x_0}, \gamma_{x_1}$ at $x_i = (z_i, 0, 0, 0)$, where z_0 and z_1 are the minimum and maximum of h. We must prove that

$$i_H(\gamma_{x_0}) = i'(C), \quad i_H(\gamma_{x_1}) = i'(C) + 1. \tag{4.4}$$

By definition $i_H(\gamma_{x_0})$ has the following property: take a map $u : I^2 \longrightarrow M$ such that $u(0,t) = u(1,t) = x_0$, $u(s,0) \in L$ and $u(s,1) \in L'$ for all s, t. Then

$$i(\gamma_{x_0}) - i_H(\gamma_{x_0}) = I_{0,H}(u).$$

u can be chosen to be the constant map at x_0. The local coordinates which we are using provide a trivialization of u^*TM. To compute $I_{0,H}(u)$ one has to choose a subbundle $F \subset u^*TM$ with certain properties: one possible choice is

$$F_{(s,t)} = D\phi_{st}^H(TL_{x_0}) = \{(r, r \cdot s \cdot t \cdot h''(z_0)) : r \in \mathbb{R}\} \times \mathbb{R} \times 0.$$

$I_{0,H}(u)$ is defined as the Maslov index of the paths $\lambda(s) = F_{(s,1)}$,

$$\lambda'(s) = TL'_{x_0} = \mathbb{R} \times 0 \times 0 \times \mathbb{R}.$$

Using the definition in [15] and the fact that $h''(z_0) > 0$, one obtains $\mu(\lambda, \lambda') = \frac{1}{2}$. Therefore

$$i_H(\gamma_{x_0}) = i(\gamma_{x_0}) - \tfrac{1}{2} = i'(C).$$

The same argument can be used to prove the second part of (4.4).

5. Geodesics

This section summarizes the classical relationship between geodesics and Lagrangian intersections. Let (P, g) be a compact Riemannian manifold and $c : I \longrightarrow P$ a geodesic. For $r \in I$, let $m(c, r) \in \mathbb{Z}$ be the multiplicity of $c(0)$ and $c(r)$ as conjugate points along c. The energy and Morse index of c are defined by

$$e(c) = \tfrac{1}{2} g(\dot{c}(0), \dot{c}(0)),$$
$$m(c) = \sum_{0<r<1} m(c, r) + \tfrac{1}{2} m(c, 1). \tag{5.1}$$

$m(c)$ is not necessarily an integer; we have adjusted the contribution of the endpoints to suit Robbin-Salamon's conventions for the Maslov index.

Let (M, ω) be the tangent bundle TP together with the symplectic form obtained by identifying $TP \cong T^*P$. Let $(\phi_t)_{t \in \mathbb{R}}$ be the geodesic flow on M, that is, the Hamiltonian flow of $H(\xi) = \frac{1}{2} g(\xi, \xi)$. Choose two points $p, p' \in P$, and consider the Lagrangian submanifolds $L = \phi_{-1}(TP_p), L' = TP_{p'} \subset M$. Their intersection points correspond to geodesics from p' to p. More precisely, a point $\xi \in L'$ lies in $N = L \cap L'$ iff the unique geodesic $c_\xi : I \longrightarrow P$ with $\dot{c}_\xi(0) = \xi$ satisfies $c_\xi(1) = p$. The numbers $m(c_\xi, r)$ can be written in terms of the derivative of ϕ:

$$m(c_\xi, r) = \dim\left(\Lambda_\xi \cap [D\phi_{-r}(\Lambda)]_\xi\right), \tag{5.2}$$

where $\Lambda \subset TM$ is the vertical part of TM, that is, the tangent bundle along the fibres of the projection $M \longrightarrow P$. In particular $m(c_\xi, 1) = \dim(TL_\xi \cap TL'_\xi)$. Therefore L and L' have clean intersection iff N is a submanifold and $\dim N = m(c_\xi, 1)$ for all $\xi \in N$; the last condition means that every Jacobi field along c_ξ with vanishing boundary values comes from a geodesic variation of c_ξ which leaves the endpoints fixed.

It is easy to see in the present case both $[\alpha] \in H^1(\mathcal{P}(L, L'); \mathbb{R})$ and the class $\chi \in H^1(\mathcal{P}(L, L'); \mathbb{Z})$ defined in the previous section vanish. Hence one can choose an action functional $a : \mathcal{P}(L, L') \longrightarrow \mathbb{R}$, and a coherent choice of indices $i(\gamma_\xi)$. Both are not unique; the following Proposition holds for one particular choice.

Proposition 5.1. *Let $\gamma_\xi \in \mathcal{P}(L, L')$ be the constant path at a point $\xi \in L \cap L'$, and c_ξ the corresponding geodesic. Then $a(\gamma_\xi) = e(c_\xi)$ and $i(\gamma_\xi) = m(c_\xi)$.*

We begin by considering a slightly more general situation.

Lemma 5.2. *Let (M, ω) be a symplectic manifold, and $L_0, L_0' \subset M$ a pair of Lagrangian submanifolds such that*

(1) there is a $\theta \in \Omega^1(M)$ with $d\theta = \omega$ and $\theta | L_0 = 0$, $\theta | L_0' = 0$;

(2) there is a Lagrangian subbundle $\Lambda \subset TM$ with $\Lambda | L_0 = TL_0$, $\Lambda | L_0' = TL_0'$.

Take a proper function $H \in C^\infty(M; \mathbb{R})$ with Hamiltonian vector field X, and let (ϕ_t) be its flow. Set $L = \phi_{-1}(L_0)$, $L' = L_0'$. Then the two classes $[\alpha] \in H^1(\mathcal{P}(L, L'); \mathbb{R})$ and $\chi \in H^1(\mathcal{P}(L, L'); \mathbb{Z})$ vanish, and for a suitable choice of action a and index i, the following holds:

(1') $a(\gamma_x) = -H(x) + \int_I (i_X \theta)(\phi_t(x))\, dt$ for all $x \in L \cap L'$.

(2') Take $x \in L \cap L'$ and choose a symplectic isomorphism $TM_x \cong \mathbb{R}^{2n}$. Then

$$i(\gamma_x) = \mu(\lambda_x, \lambda_x') - \tfrac{1}{2} \dim L, \tag{5.3}$$

where $\lambda_x, \lambda_x' : I \longrightarrow \mathcal{L}(n)$ are given by $\lambda_x(r) = \Lambda_x$ and $\lambda_x'(r) = [D\phi_{-r}(\Lambda)]_x$.

Proof. We prove only the statement (2') and leave the rest to the reader. Take two points $x_-, x_+ \in L \cap L'$ and a map $u : [a; b] \times I \longrightarrow M$ which corresponds to a path from γ_{x_-} to γ_{x_+} in $\mathcal{P}(L, L')$. In order to compute $I(u)$ one has to choose a trivialization of $E = u^*TM$ and a Lagrangian subbundle $F \subset E$ with certain properties. One suitable choice is $F_{(s,t)} = [D\phi_{-1}(\Lambda)]_{u(s,t)}$. $I(u)$ is the Maslov index of the pair (λ, λ') given by

$$\lambda(s) = F_{(s,1)} = [D\phi_{-1}(\Lambda)]_{u(s,1)}, \quad \lambda'(s) = TL'_{(s,1)} = \Lambda_{u(s,1)}.$$

Consider another Lagrangian subbundle $F' \subset E$ defined by

$$F'_{(s,t)} = [D\phi_{s-1}(\Lambda)]_{u(s,t)}.$$

For any path α in $[a; b] \times I$, we denote by $\tilde{\mu}(\alpha)$ the Maslov index of (F, F') along α, that is, the Maslov index of $r \longmapsto (F_{\alpha(r)}, F'_{\alpha(r)})$. For instance, the expression for $I(u)$ given above says that $I(u) = \tilde{\mu}(\alpha_2)$ where $\alpha_2 : [a; b] \longrightarrow [a; b] \times I$ is the path $\alpha_2(r) = (r, 1)$. Now take the other three sides of the boundary of $[a;b]\times I$: $\alpha_1(r)=(r,0)$, $(a\le r\le b)$ and $\alpha_3(r)=(a,r)$, $\alpha_4(r)=(b,r)$, $(0\le r\le 1)$. Because of the additivity and homotopy invariance of the Maslov index for paths,

$$I(u)=\tilde\mu(\alpha_2)=-\tilde\mu(\alpha_3)+\tilde\mu(\alpha_1)+\tilde\mu(\alpha_4). \tag{5.4}$$

Since F and F' agree over $[a;b]\times 0$, $\tilde\mu(\alpha_1)=0$. $\tilde\mu(\alpha_3)$ and $\tilde\mu(\alpha_4)$ are independent of u; they depend only on x_- and x_+, respectively. After changing the trivialization of E by $D\phi_{1-s}$ one sees that $\tilde\mu(\alpha_3)=\mu(\lambda'_{x_-},\lambda_{x_-})$, and therefore (by property (iv) of μ) $-\tilde\mu(\alpha_3)=\mu(\lambda_{x_-},\lambda'_{x_-})$. Similarly, $-\tilde\mu(\alpha_4)=\mu(\lambda_{x_+},\lambda'_{x_+})$. Equation (5.4) implies that (5.3) is a coherent choice of indices. The constant $\frac{1}{2}\dim L$ has been subtracted in order to fulfil the integrality criterion (4.1). q.e.d.

Proof of Proposition 5.1. Let $\theta\in\Omega^1(M)$ be the form corresponding to the canonical one-form on T^*P, and $\Lambda\subset TM$ the vertical subbundle, and let $H(\xi)=\frac{1}{2}g(\xi,\xi)$. $L_0=TP_p$ and $L'_0=TP_{p'}$ satisfy the conditions of Lemma 5.2. Using the first part of that Lemma and the fact that $i_X\theta=2H$, one obtains

$$a(\gamma_\xi)=-H(\xi)+\int_I(i_X\theta)(\phi_t(x))\,dt=H(\xi)$$

for any $\xi\in L\cap L'$. Choose a symplectic isomorphism $TM_\xi\cong\mathbb{R}^{2n}$ induced by an isomorphism $TP_{p'}\cong\mathbb{R}^n$ and by the Levi-Civita connection. Then the paths λ_ξ,λ'_ξ defined in Lemma 5.2 are of the following form: $\lambda_\xi(r)=\mathbb{R}^n\times 0$, and $\lambda'_\xi(r)=A(r)^{-1}(\mathbb{R}^n\times 0)$, where $A:[0;1]\longrightarrow \mathrm{Sp}(2n,\mathbb{R})$ satisfies a differential equation

$$\dot A(r)=\begin{pmatrix}0 & R(r)\\ \mathbf{1} & 0\end{pmatrix}A(r),\quad A(0)=\mathbf{1} \tag{5.5}$$

for some family $R(r)$ of symmetric $n\times n$-matrices obtained from the curvature tensor of (P,g). This is just the equation for Jacobi fields, written as a first order equation. In view of (5.1) and (5.2), the proof of Proposition 5.1 is completed by applying the following property of the Maslov index for paths:

Lemma 5.3. *Let $R(r)$, $0\le r\le 1$, be a family of symmetric $n\times n$ matrices, and let $A(r)$ be the solution of* (5.5). *Consider paths*

$$\lambda,\lambda':[0;1]\longrightarrow\mathcal{L}(n)$$

given by

$$\lambda(r) = \mathbb{R}^n \times 0, \quad \lambda'(r) = A(r)^{-1}(\mathbb{R}^n \times 0).$$

Then their Maslov index is

$$\mu(\lambda, \lambda') = \tfrac{1}{2}\dim(\lambda(0) \cap \lambda'(0)) + \sum_{0<r<1} \dim(\lambda(r) \cap \lambda'(r)) + \tfrac{1}{2}\dim(\lambda(1) \cap \lambda'(1)).$$

This property can be deduced easily from the definition of μ given in [15].

6. Generalized Dehn twists

This section contains the definition of the maps τ_L. The following elementary fact will be used several times:

Lemma 6.1. *Let (M, ω) be a symplectic manifold, $H \in C^\infty(M, \mathbb{R})$ and $\Psi \in C^\infty(\mathbb{R}, \mathbb{R})$. Then the Hamiltonian flows of H and $\Psi(H)$ are related by*

$$\phi_t^{\Psi(H)}(x) = \phi^H_{t\Psi'(H(x))}(x).$$

Let η be the standard symplectic form on T^*S^2, and $S^2 \subset T^*S^2$ the zero-section. Its complement $T^*S^2 \setminus S^2$ carries a Hamiltonian circle action σ with moment map $\mu(\xi) = |\xi|$ (the length function with respect to the standard metric). To see that this is a circle action, recall that if we identify $T^*S^2 = TS^2$, then the flow induced by $\frac{1}{2}\mu^2$ is the geodesic flow. By Lemma 6.1, μ itself induces the *normalized geodesic flow* which transports any nonzero tangent vector ξ with unit speed along the geodesic emanating from it, irrespective of what $|\xi|$ is. Since all geodesics of length 2π are closed, this is a circle action. σ does not extend continuously over the zero-section, with one exception: since any geodesic of length π on S^2 connects two opposite points, $\sigma(-1)$ is the restriction of the antipodal involution $A : T^*S^2 \longrightarrow T^*S^2$ to $T^*S^2 \setminus S^2$.

Take a function $\psi \in C^\infty(\mathbb{R}, \mathbb{R})$ such that $\psi(t) + \psi(-t) = 2\pi$ for all t, and $\psi(t) = 0$ for $t \gg 0$. Let $\tau : T^*S^2 \longrightarrow T^*S^2$ be the map defined by

$$\tau(\xi) = \begin{cases} \sigma(e^{i\psi(|\xi|)})(\xi), & \xi \notin S^2, \\ A(\xi), & \xi \in S^2. \end{cases}$$

τ is smooth and symplectic, and it is the identity outside a compact subset. The third property is obvious; to prove the first two, consider

$$(A \circ \tau)(\xi) = \sigma(e^{i(\psi(|\xi|)-\pi)})(\xi).$$

Take a function $\Psi \in C_c^\infty(\mathbb{R}, \mathbb{R})$ with $\Psi'(t) = \psi(t) - \pi$. Since $\psi - \pi$ is odd, Ψ is even, and hence $\xi \longmapsto \Psi(|\xi|)$ is smooth on all of T^*S^2. Lemma 6.1 shows that $A \circ \tau$ is the time-one map of the Hamiltonian flow of $\Psi(|\xi|)$. In particular, it is smooth and symplectic, and therefore so is τ. We call τ a *model generalized Dehn twist.*

Let (M, ω) be a symplectic four-manifold containing a Lagrangian two-sphere L. By a theorem of Weinstein, there is a symplectic embedding $f : D_\epsilon(T^*S^2) \longrightarrow M$ of the disc bundle $D_\epsilon(T^*S^2) = \{\xi \in T^*S^2 \mid |\xi| < \epsilon\}$ into M, for some $\epsilon > 0$, such that $f(S^2) = L$. Let τ be the model generalized Dehn twist associated to a function ψ such that $\psi(t) = 0$ for all $t > \epsilon/2$. Then one can define a symplectic automorphism τ_L of M by

$$\tau_L(x) = \begin{cases} f\tau f^{-1}(x), & x \in \mathrm{im}(f), \\ x, & \text{otherwise.} \end{cases}$$

We call such a map τ_L a generalized Dehn twist along L.

Lemma 6.2. *The symplectic isotopy class of τ_L is independent of the choice of f and ψ. Moreover, if L and L' are Lagrangian isotopic, then τ_L and $\tau_{L'}$ are symplectically isotopic.*

The independence of ψ can be proved by an explicit isotopy. Next, consider two embeddings $f, f' : D_\epsilon(T^*S^2) \longrightarrow M$ with $f(S^2) = f'(S^2) = L$. If f can be deformed to f' through symplectic embeddings which map S^2 to L, then the corresponding generalized Dehn twists are symplectically isotopic. The same holds if f can be deformed to f' after making ϵ smaller. Such a deformation of the germs of f, f' exists iff the restrictions $f|S^2, f'|S^2 : S^2 \longrightarrow L$ are differentiably isotopic. Since $\mathrm{Diff}^+(S^2)$ is path-connected, this holds iff f and f' induce the same orientation of L. To complete the proof that τ_L is independent of the choice of embedding, it is enough to find two examples f, f', which induce opposite orientations of L but define the same generalized Dehn twist, and that is easy: take an arbitrary f and set $f' = f \circ A$. Finally, it is clear that the symplectic isotopy class of τ_L depends on L only up to ambient symplectic isotopy. However, that is the same as Lagrangian isotopy.

An inspection of the proof which we have just given shows that τ_L is well-defined up to Hamiltonian isotopy. We do not need this sharper statement here.

Lemma 6.3. *Let τ_L be a generalized Dehn twist along a Lagrangian two-sphere L. Then the square τ_L^2 is differentiably isotopic to the identity.*

Proof. We use the model

$$T^*S^2 = \{(u,v) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid |u| = 1 \text{ and } \langle u,v \rangle = 0\},$$

in which $\eta = \sum_i dv_i \wedge du_i$. For $x \in \mathbb{R}^3 \setminus 0$ and $t \in \mathbb{R}$, let $R^t(x) \in SO(3)$ be the rotation with axis $x/|x|$ and angle t. Then

$$\sigma(e^{it})(u,v) = (R^t(u \times v)u, R^t(u \times v)v).$$

Consider the following one-parameter family $\sigma^{(s)}$, $0 \leq s \leq 1$, of smooth circle actions on $T^*S^2 \setminus S^2$:

$$\sigma^{(s)}(e^{it})(u,v) = (R^t(su + (1-s)u \times v)u, R^t(su + (1-s)u \times v)v).$$

$\sigma^{(0)} = \sigma$. On the other hand, $\sigma^{(1)}$ is the action of S^1 by rotation in each fibre of T^*S^2 and extends smoothly to the zero-section S^2. The square of a model generalized Dehn twist is

$$\tau^2(\xi) = \begin{cases} \sigma(e^{2i\psi(|\xi|)})(\xi), & \xi \notin S^2, \\ \xi, & \xi \in S^2. \end{cases}$$

We can assume that $\psi(t) = \pi$ for small $|t|$; then τ^2 is the identity in a neighbourhood of the zero-section. Replacing σ by $\sigma^{(s)}$ defines a differentiable isotopy from τ^2 to $T(\xi) = \sigma^{(1)}(e^{2i\psi(|\xi|)})(\xi)$, and this can be deformed to the identity by changing ψ to $s\psi$ for $0 \leq s \leq 1$. This isotopy from τ^2 to the identity is local in the sense that if $\tau = \text{id}$ outside $D_\epsilon(T^*S^2)$ for some $\epsilon > 0$, then the same holds for the isotopy. This implies the Lemma as stated. q.e.d.

Remarks 6.4.

1. Let $L \subset M$ be a Lagrangian two-sphere and f a symplectic automorphism of M. It follows immediately from the definition of generalized Dehn twists that $\tau_{f(L)} = f\tau_L f^{-1}$.

2. The definition of a generalized Dehn twist is sensitive to the sign of ω. A generalized Dehn twist along L as a submanifold of $(M, -\omega)$ is the inverse of a generalized Dehn twist along $L \subset (M, \omega)$.

3. Let $\mathrm{Aut}^c(T^*S^2, \eta)$ be the group of those symplectic automorphisms of T^*S^2 which are equal to the identity outside a compact subset, and $[\tau] \in \pi_0(\mathrm{Aut}^c(T^*S^2, \eta))$ the class containing all model generalized Dehn twists. Corollary 1.2 implies that $[\tau]$ has infinite order. It can be shown [20] that $[\tau]$ generates $\pi_0(\mathrm{Aut}^c(T^*S^2, \eta))$, and that the higher homotopy groups are trivial.

4. The definition of a model generalized Dehn twists extends in a straightforward way to the cotangent bundle of S^n for all n. Using this as a local model, one can define generalized Dehn twists associated to Lagrangian embeddings of S^n into $2n$-dimensional symplectic manifolds. For $n = 1$ these are just the ordinary positive Dehn twists along a curve on a surface.

7. Proof of Theorem 1.1

Let (M, ω) and L_1, L_2, L_3 be as in that Theorem. Fix some $r \in \mathbb{N}$. One can find a symplectic embedding $f : D_\epsilon(T^*S^2) \longrightarrow M$ for some $\epsilon > 0$, such that $f(S^2) = L_2$, $f^{-1}(L_1) = T^*_xS^2 \cap D_\epsilon(T^*S^2)$ and $f^{-1}(L_3) = T^*_{A(x)}S^2 \cap D_\epsilon(T^*S^2)$ for some $x \in S^2$. After rescaling ω if necessary, one can assume that $\epsilon = 2\pi r$. Let τ_{L_2} be the generalized Dehn twist along L_2 defined using the embedding f and some function ψ, and $L_1^{(r)} = \tau_{L_2}^{2r}(L_1)$. $L_1^{(r)} \cap L_3$ is contained in $\mathrm{im}(f)$, and $f^{-1}(L_1^{(r)} \cap L_3) = \tau^{2r}(T^*_xS^2) \cap T^*_{A(x)}S^2 = \{\xi \in T^*_{A(x)}S^2 \mid 2r\psi(|\xi|) + \pi \in 2\pi\mathbb{Z}\}$, where τ is the model generalized Dehn twist determined by ψ. Now assume that ψ satisfies

$$\begin{cases} \psi'(t) \leq 0 & \text{for all } t, \\ \psi(t) = \pi - t/2r & \text{for } 0 \leq t \leq \delta = 2\pi(r - \frac{1}{4}), \\ \psi(t) = 0 & \text{for } t \geq 2\pi r. \end{cases}$$

Then $L_1^{(r)} \cap L_3$ is the disjoint union of r circles $C_1, \dots, C_r$, where

$$f^{-1}(C_j) = \{\xi \in T^*_{A(x)}S^2 \mid \psi(|\xi|) = \tfrac{2j-1}{2r}\pi\}.$$

Note that all C_j lie in $f(U)$, where $U = D_\delta(T^*S^2) \subset D_\epsilon(T^*S^2)$. This is important because $\tau^{2r}(\xi) = \sigma(e^{-i|\xi|})(\xi)$ for all $\xi \in U$. Since σ is defined by normalizing the geodesic flow (ϕ_t), this means that $\tau^{2r}|U = \phi_{-1}|U$. Setting $L = \phi_{-1}(T^*_x S^2)$ and $L' = T^*_{A(x)} S^2$, we have shown that

$$f^{-1}(L_1^{(r)}) \cap U = L \cap U, \quad f^{-1}(L_3) \cap U = L' \cap U,$$

which makes it possible to apply the results of section 5. First of all, the intersection points of $L_1^{(r)}$ and L_3 are the intersection points of L and L' inside U, and these correspond to geodesics from $A(x)$ to x of length $\leq \delta$. More precisely, the circle C_j corresponds to the one-parameter family of geodesics which wind $j - \frac{1}{2}$ times around S^2. In section 5 we have given a criterion, in terms of Jacobi fields, for L and L' to have clean intersection. This is satisfied in the present case. Hence $L_1^{(r)}$ and L_3 also have clean intersection. To compute the relative action and index of two intersection points $x_-, x_+ \in L_1^{(r)} \cap L_3$ one can use a path in $\mathcal{P}(L_1^{(r)}, L_3)$ whose image lies inside $f(U)$. Therefore the relative action and index coincide with those of $f^{-1}(x_-), f^{-1}(x_+)$ as intersection points of L and L'. Using Proposition 5.1, one obtains that the action $a_j \in \mathbb{R}$ of a constant path at a point of C_j satisfies

$$a_j - a_{j-1} = \tfrac{\pi^2}{2}((2j-1)^2 - (2j-3)^2) > 0.$$

The Morse index of a geodesic from $A(x)$ to x which winds $j - \frac{1}{2}$ times around S^2 is $2j - \frac{3}{2}$ (it has $2j$ conjugate points on it, including both endpoints, and all of them have multiplicity one; according to our definition, one endpoint does not contribute at all, while the other contributes $\frac{1}{2}$). Therefore

$$i'(C_j) - i'(C_{j-1}) = 2.$$

This completes the computations necessary to apply Pozniak's spectral sequence, as described in section 2. Note that since $L_1^{(r)}$ and L_3 are orientable and intersect in a union of circles, we have provided proofs of the basic results underlying the spectral sequence (see Proposition 3.2 and the discussion following Theorem 4.1).

8. A family of examples

A *configuration of Lagrangian two-spheres* in a symplectic four-manifold is a finite collection of Lagrangian two-spheres any two of which intersect transversely. An (A_m)-configuration, for $m \geq 1$, consists of m Lagrangian two-spheres $L_1, \dots, L_m$ such that

$$|L_i \cap L_j| = \begin{cases} 1, & i - j = \pm 1, \\ 0, & |i - j| \geq 2. \end{cases} \tag{8.1}$$

Proposition 8.1. *Let (H, ω) be the affine hypersurface $z_1^2 + z_2^2 = z_3^{m+1} + \frac{1}{2}$ in $\mathbb{C}^3$, equipped with the standard symplectic form. For any m, (H, ω) contains an (A_m)-configuration of Lagrangian two-spheres.*

Proof. The projection $\pi : H \longrightarrow \mathbb{C}^2$ onto (z_1, z_2) is an $(m+1)$-fold covering branched along $C = \{z_1^2 + z_2^2 = \frac{1}{2}\} \subset \mathbb{C}^2$. The covering group is generated by $\sigma(z_1, z_2, z_3) = (z_1, z_2, e^{2\pi i/(m+1)} z_3)$. Let ω_0 be the standard symplectic form on $\mathbb{C}^2$.

Lemma 8.2. *Let $K \subset \mathbb{C}^2 \setminus C$ be a compact subset. There is a symplectic form ω' on H which is diffeomorphic to ω and such that*

$$\omega' | \pi^{-1}(U) = \pi^*(\omega_0 | U) \tag{8.2}$$

for some neighbourhood $U \subset \mathbb{C}^2$ of K.

Proof of Lemma 8.2. K is contained in the open subset

$$U = \{z \in \mathbb{C}^2 \mid \epsilon < |z_1^2 + z_2^2 - \tfrac{1}{2}|^{1/(m+1)} < \epsilon^{-1}\}$$

for sufficiently small $\epsilon > 0$. Choose a function $\beta \in C^\infty(\mathbb{R}^{\geq 0}, \mathbb{R})$ such that $\beta(r) \leq 1$ for all r, $\beta(r) = 0$ for $r \leq \epsilon/2$ or $r \geq 2\epsilon^{-1}$, $\beta(r) = 1$ for $\epsilon \leq r \leq \epsilon^{-1}$, and $\int_0^\infty r\beta(r)\, dr = 0$. Set

$$\omega' = \omega - \beta(|z_3|)(\tfrac{i}{2} dz_3 \wedge d\bar{z}_3).$$

By definition ω' satisfies (8.2). Moreover, it is a symplectic form which is compatible with the complex structure; it agrees with ω outside a compact subset; and the difference $\omega' - \omega$ represents the trivial class in $H^2_c(H; \mathbb{R})$. By a familiar argument it follows that ω and ω' are diffeomorphic. q.e.d.

Now consider the two-dimensional figure-eight map

$$\mathbb{R}^3 \supset S^2 \xrightarrow{f} \mathbb{C}^2 \setminus C, \quad f(t_1, t_2, t_3) = (t_2(1 + it_1), t_3(1 + it_1)).$$

f is an immersion with one double point $0 = f(\pm 1, 0, 0)$ at which the two branches meet transversely. Moreover, if $\gamma : [0; 1] \longrightarrow S^2$ is any path from $(1,0,0)$ to $(-1,0,0)$, $f(\gamma)$ is a loop in $\mathbb{C}^2 \setminus C$ whose linking number with C equals one. Let $\tilde{f} : S^2 \longrightarrow H$ be a lift of f to H; such a lift exists because f avoids the branch locus of π. The fact that $f(\gamma)$ has linking number 1 with C implies that

$$\tilde{f}(-1,0,0) = \sigma(\tilde{f}(1,0,0)). \tag{8.3}$$

Therefore $\tilde{f}$ is an embedding. Now consider the shifted embedding $\sigma \circ \tilde{f}$. If $m \geq 2$, the images of $\tilde{f}$ and of $\sigma \circ \tilde{f}$ do not have any intersection points except for (8.3). The intersection at that point is modelled on the self-intersection of f; hence it is transverse. A repetition of the same argument shows that $L_1 = \mathrm{im}(\tilde{f}), L_2 = \sigma(L_1), \dots, L_m = \sigma^{m-1}(L_1)$ from a family of smoothly embedded two-spheres which intersect according to (8.1). Take a symplectic form ω' as in Lemma 8.2 with $K = \mathrm{im}(f)$. Since f is a Lagrangian immersion with respect to ω_0, the submanifolds $L_1, \dots, L_m$ are ω'-Lagrangian. This proves that (H, ω') contains an (A_m)-configuration. Since ω' is diffeomorphic to ω, it follows that (H, ω) contains one as well. q.e.d.

Remark 8.3.

1. By leaving out some components, one sees that (H, ω) contains an (A_3)-configuration whenever $m \geq 3$. This configuration lies in the bounded subset (1.1) if R is sufficiently large. Taking R large also ensures that the closure of (1.1) is a symplectic manifold with contact type boundary. Both the symplectic class and the first Chern class of (1.1) vanish, because it is an open subset of an affine hypersurface. Hence (1.1) satisfies the conditions of Theorem 1.1.

2. The existence of m smooth embedded two-spheres in H satisfying (8.1) is a consequence of Brieskorn's resolution [3]. The only new aspect of Proposition 8.1 is that one can choose these spheres to be Lagrangian.

3. A straightforward generalization of the proof given above produces an (A_m)-configuration of Lagrangian n-spheres in the hypersurface $z_1^2 + z_2^2 + \cdots + z_n^2 = z_{n+1}^{m+1} + \frac{1}{2}$ for any m, n.

Appendix A. Lagrangian surgery

The aim of this Appendix is to relate generalized Dehn twists to the

Lagrangian surgery construction which has been studied by Polterovich [13] and others. As a by-product we obtain the following result:

Proposition 8.4. *Let L_1 and L_2 be two Lagrangian two-spheres in a symplectic four-manifold (M,ω). Assume that they intersect transversely in a single point. Then $\tau_{L_1}\tau_{L_2}\tau_{L_1}$ and $\tau_{L_2}\tau_{L_1}\tau_{L_2}$ are symplectically isotopic automorphisms.*

In particular, an (A_m)-configuration in a symplectic four-manifold defines a homomorphism from the braid group B_{m+1} to the group of symplectic isotopy classes of automorphisms of the manifold. This holds e.g. for the manifolds (1.1).

We begin by recalling the definition of Lagrangian surgery. Our exposition follows [13] with some modifications. Let $C \subset \mathbb{R}^2$ be a smooth embedded curve with the following properties: C is diffeomorphic to $\mathbb{R}$; it coincides with $(\mathbb{R}^+ \times 0) \cup (0 \times \mathbb{R}^-)$ outside a compact subset; and there is no $x \in \mathbb{R}^2$ such that both x and $-x$ lie in C. Consider

$$H = \{(y_1 \cos t, y_1 \sin t, y_2 \cos t, y_2 \sin t) \mid (y_1, y_2) \in C,\ t \in S^1\} \subset \mathbb{R}^4.$$

H is an embedded surface diffeomorphic to $\mathbb{R} \times S^1$; it is Lagrangian with respect to $\omega = dx_1 \wedge dx_3 + dx_2 \wedge dx_4$; and it coincides with $(\mathbb{R}^2 \times 0) \cup (0 \times \mathbb{R}^2)$ outside a compact subset. By choosing C suitably, one can arrange that the last-mentioned property holds outside an arbitrarily small neighbourhood of $0 \in \mathbb{R}^4$. H is called a *Lagrangian handle*.

Now let (M,ω) be a symplectic four-manifold and $L_1, L_2 \subset M$ two compact Lagrangian surfaces which intersect transversely and in a single point x. Choose a neighbourhood $U \subset \mathbb{R}^4$ of 0 and a Darboux chart $f : U \longrightarrow M$ such that $f(0) = x$, $f^{-1}(L_1) = (\mathbb{R}^2 \times 0) \cap U$ and $f^{-1}(L_2) = (0 \times \mathbb{R}^2) \cap U$. Let H be a Lagrangian handle which agrees with $(\mathbb{R}^2 \times 0) \cup (0 \times \mathbb{R}^2)$ outside U. Define a new Lagrangian submanifold $L \subset M$ by $L \cap f(U) = f(H)$ and $L \setminus f(U) = (L_1 \cup L_2) \setminus f(U)$. L is diffeomorphic to the connected sum of L_1 and L_2. It is called the Lagrangian surgery of L_1 and L_2; we denote it by $L_1 \# L_2$. One can show that this surgery is independent of all choices up to Lagrangian isotopy.

Proposition 8.5. *Assume that L_2 is a Lagrangian sphere. Then $L_1 \# L_2$ is Lagrangian isotopic to $\tau_{L_2}^{-1}(L_1)$.*

Proof. Let τ be the model generalized Dehn twist on T^*S^2 defined using a function ψ such that $\psi' \leq 0$ everywhere, and

$$\psi(t) = \pi - t \text{ for } t \leq \epsilon, \quad \psi(t) > 0 \text{ for } \epsilon < t < 2\epsilon, \quad \psi(t) = 0 \text{ for } t \geq 2\epsilon$$

for some $\epsilon > 0$. Choose a point $x \in S^2$, and set $L = \tau^{-1}(T_x^* S^2)$. The exponential maps at x and $A(x)$ induce symplectic isomorphisms

$$f_x : T^*B_\pi \longrightarrow T^*S^2 \setminus T^*_{A(x)}S^2,$$
$$f_{A(x)} : T^*B_\pi \longrightarrow T^*S^2 \setminus T^*_x S^2.$$

Here $B_\pi \subset \mathbb{R}^2$ is the open ball of radius π. We will identify T^*B_π with $\mathbb{R}^2 \times B_\pi$. In these coordinates

$$f_x^{-1}(L) = \{(p, -\tfrac{\psi(|p|)}{|p|}p) \mid p \in \mathbb{R}^2 \setminus 0\},$$
$$f_{A(x)}^{-1}(L) = \{(p, -\tfrac{\pi-\psi(|p|)}{|p|}p) \mid p \in B_{2\epsilon}\}.$$

Let $V = B_\epsilon \times B_\epsilon \subset T^*B_\pi$. Then

$$f_{A(x)}^{-1}(L) \cap V = \{(p, -p) \mid p \in B_\epsilon\}.$$

It follows that L can be deformed (by a symplectic isotopy which is trivial outside $f_{A(x)}(V)$) into the Lagrangian submanifold $L' \subset T^*S^2$ defined by

$$f_x^{-1}(L') = \{(\rho(\pi - \psi(|p|))p, -\tfrac{\psi(|p|)}{|p|}p) \mid p \in \mathbb{R}^2 \setminus 0\},$$
$$f_{A(x)}^{-1}(L') = \{(\rho(|p|)p, -\tfrac{\pi-\psi(|p|)}{|p|}p) \mid p \in B_{2\epsilon}\}.$$

Here $\rho \in C^\infty(\mathbb{R}^{\geq 0}, \mathbb{R})$ is a cutoff function with $\rho(t) = 0$ for $t \leq \epsilon/4$ and $\rho(t) = 1$ for $t \geq \epsilon/2$. Note that L' agrees with $T_x^*S^2 \cup S^2$ outside $f_x(W)$, where $W = B_{2\epsilon} \times B_{\pi-\epsilon/4}$. The remaining portion of L' can be written as

$$f_x^{-1}(L') \cap W = \{(y_1 \cos t, y_1 \sin t, y_2 \cos t, y_2 \sin t) \mid (y_1, y_2) \in C,\ t \in S^1\} \cap W,$$

where $C \subset \mathbb{R}^2$ is the image of the embedding

$$c : \mathbb{R}^+ \longrightarrow \mathbb{R}^2, \qquad c(t) = (\rho(\pi - \psi(t))t, -\psi(t)).$$

This is just the essential part of a Lagrangian handle in $\mathbb{R}^4$.

Given two compact Lagrangian surfaces $L_1, L_2 \subset M$ which intersect transversely in a single point x and such that L_2 is a Lagrangian two-sphere, one can always find a symplectic embedding

$$f : D_\epsilon(T^*S^2) \longrightarrow M,$$

for some $\epsilon > 0$, such that $f(S^2) = L_2$ and

$$f^{-1}(L_1) = T_x^* S^2 \cap D_\epsilon(T^*S^2)$$

for some $x \in S^2$. The argument above then proves that $\tau_{L_2}^{-1}(L_1)$ is Lagrangian isotopic to $L_1 \# L_2$. q.e.d.

The same argument shows that

Proposition 8.6. *Assume that L_1 is a Lagrangian sphere. Then $L_1 \# L_2$ is Lagrangian isotopic to $\tau_{L_1}(L_2)$.*

Proof of Proposition 8.4. Set $L_2' = \tau_{L_1}(L_2)$ and $L_1' = \tau_{L_2}^{-1}(L_1)$. As a special case of Remark 6.41, $\tau_{L_1}\tau_{L_2}\tau_{L_1}^{-1}$ is symplectically isotopic to $\tau_{L_2'}$. Similarly $\tau_{L_2}^{-1}\tau_{L_1}\tau_{L_2}$ is symplectically isotopic to $\tau_{L_1'}$. Propositions 8.5 and 8.6 show that L_1' and L_2' are both Lagrangian isotopic to $L_1 \# L_2$ and hence Lagrangian isotopic to each other. It follows that $\tau_{L_1'}$ is symplectically isotopic to $\tau_{L_2'}$. q.e.d.

References

[1] V. I. Arnol'd, *Some remarks on symplectic monodromy of Milnor fibrations*, The Floer Mem. Vol. (H. Hofer, C. Taubes, A. Weinstein, and E. Zehnder, eds.), Progr. Math., Vol. 133, Birkhäuser, Boston, MA, 1995, 99–104.

[2] D. Austin & P. Braam, *Morse-Bott theory and equivariant cohomology*, The Floer Mem. Vol. (H. Hofer, C. Taubes, A. Weinstein, and E. Zehnder, eds.), Progr. Math., Vol. 133, Birkhäuser, Boston, MA, 1995, 123–183.

[3] E. Brieskorn, *Über die Auflösung gewisser Singularitäten von holomorphen Abbildungen*, Math. Ann. **166** (1966) 76–102.

[4] Ya. Eliashberg & L. Polterovich, *The problem of Lagrangian knots in four-manifolds*, Geom. Topology. Proc. 1993 Georgia Internat. Topology Conference (W. H. Kazez, ed.), Internat. Press, 1997, 313–327.

[5] A. Floer, *Morse theory for Lagrangian intersections*, J. Differential Geom. **28** (1988) 513–547.

[6] ———, *A relative Morse index for the symplectic action*, Comm. Pure Appl. Math. **41** (1988) 393–407.

[7] ———, *Witten's complex and infinite dimensional Morse theory*, J. Differential Geom. **30** (1989) 207–221.

[8] A. Floer, H. Hofer & D. Salamon, *Transversality in elliptic Morse theory for the symplectic action*, Duke Math. J. **80** (1995) 251–292.

[9] P. Kronheimer, *Some non-trivial families of symplectic structures*, Preprint, 1997.

[10] Y.-G. Oh, *On the structure of pseudo-holomorphic discs with totally real boundary conditions*, Preprint (revised version, May 1996).

[11] ———, *Floer cohomology of Lagrangian intersections and pseudo-holomorphic discs I*, Comm. Pure Appl. Math. **46** (1993) 949–994.

[12] ———, *Floer cohomology, spectral sequences, and the Maslov class of Lagrangian embeddings*, Internat. Math. Res. Notices (1996) 305–346.

[13] L. Polterovich, *Surgery of Lagrange submanifolds*, Geom. Funct. Anal. **1** (1991) 198–210.

[14] M. Poźniak, *Floer homology, Novikov rings and clean intersections*, Ph.D. thesis, University of Warwick, 1994.

[15] J. Robbin & D. Salamon, *The Maslov index for paths*, Topology **32** (1993) 827–844.

[16] ———, *The spectral flow and the Maslov index*, Bull. London Math. Soc. **27** (1995) 1–33.

[17] Y. Ruan & G. Tian, *Bott-type symplectic Floer cohomology and its multiplication structures*, Math. Research Letters **2** (1995) 203–219.

[18] D. Salamon & E. Zehnder, *Morse theory for periodic solutions of Hamiltonian systems and the Maslov index*, Comm. Pure Appl. Math. **45** (1992) 1303–1360.

[19] P. Seidel, *Floer homology and the symplectic isotopy problem*, Ph.D. thesis, Oxford University, 1997.

[20] ———, *Symplectic automorphisms of T^*S^2*, Preprint, 1998.

[21] C. Viterbo, *Intersection des sous-variétés Lagrangiennes, fonctionelles d'action et indice des systèmes Hamiltoniens*, Bull. Soc. Math. France **115** (1987) 361–390.

Centre de Mathématiques, Ecole Polytechnique,
Palaiseau, France